Contents :

Abstract

Differential Quadrature Method proposed here can be used to solve boundary-value and initial-value differential equations with a linear or nonlinear nature.

Unlike the classic Differential Quadrature Method (DQM), the newly proposed Differential Quadrature chooses the function values and some derivatives wherever necessary as independent variables.

The δ-type grid arrangement used in the classic DQM is exempt at present work while the boundary conditions are applied exactly.

Most importantly, the weighting coefficients can be obtained explicitly using the proposed procedures.

Introduction

The Differential Quadrature Method (DQM) was proposed by Bellman and Casti (1971) and has been employed in the solution of solid mechanics problems by Bert and Malik (1996).

A δ-point technique has been employed in the DQM's application to boundary value differential equations with multiple conditions. But the initial-value differential quadrature method for structural dynamics has not been reported until now.

The classic Differential Quadrature Method (DQM)

Consider a one-dimensional variable

$$\psi(x)$$

and let $\psi_i = \psi(x_i)$ be the function values specified in a finite set of N discrete points x_i(i = 1, 2, ... ,N)

Essential basis of the DQM

Let the value of the function derivative be expressed as a linearly weighted sum of the function values.

$$\psi^{(r)}(x_i) = \frac{\mathrm{d}^r \psi(x_i)}{\mathrm{d}x^r} = \sum_{j=1}^{N} A_{ij}^{(r)} \psi_j \quad (i = 1, 2, \ldots, N) \qquad (1)$$

The weighting coefficients may be determined by some appropriate functional approximations.

The approximate functions are referred to as test functions.

Although there can be many choices of the test functions, a convenient and most commonly used choice in one-dimensional problems is the Lagrangian interpolation shape functions $l_j(x)$ where:

$$\psi(x) = \sum_{j=1}^{N} l_j(x)\psi_j \qquad (2)$$

Note that in the classic DQ method the number of test functions is equal to the number of the sampling points.

The polynomial-test-function-based weighting coefficients

1) The accuracy of differential quadrature solution depends on the accuracy of the weighting coefficients.

2) Considering the Lagrangian polynomial as the test function the weighting coefficients of rth-order derivatives will be :

$$A_{ij}^{(r)} = \frac{d^r}{dx^r} l_j(x_i) \quad (i, j = 1, 2, \ldots, N) \tag{3}$$

where:

$$l_j(x) = \frac{\phi(x)}{(x - x_j)\phi^{(1)}(x_j)}; \quad \phi(x) = \prod_{m=1}^{N} (x - x_m);$$

$$\phi^{(1)}(x_j) = \frac{d\phi(x_j)}{dx} = \prod_{m=1;m\neq j}^{N} (x_j - x_m)$$

x_i's are the locations of the grid points & N is the number of sampling points.

Using Eqs. (1), (2), and (3) based on Lagrangian interpolation test function the weighting coefficients will be derived as below:

$$A_{ij}^{(1)} = \frac{\mathrm{d}l_j(x_i)}{\mathrm{d}x} = \frac{\phi^{(1)}(x_i)}{(x_i - x_j)\phi^{(1)}(x_j)}$$

$$(i, j = 1, 2, \ldots N; \quad i \neq j)$$

$$A_{ij}^{(r)} = \frac{\mathrm{d}^r l_j(x_i)}{\mathrm{d}x^r} = r\left(A_{ii}^{(r-1)} A_{ij}^{(1)} - \frac{A_{ij}^{(r-1)}}{(x_i - x_j)}\right)$$

$$(i, j = 1, 2, \ldots N; \quad i \neq j; \quad r \geq 2)$$

$$A_{ii}^{(r)} = \frac{\mathrm{d}^r l_i(x_i)}{\mathrm{d}x^r} = -\sum_{j=1; i \neq j}^{N} A_{ij}^{(r)} (i = 1, 2, \ldots, N; \quad r \geq 1) \qquad (4)$$

Normalizing sample points domain

A convenient and natural choice for the sampling points is that of the equally spaced sampling points. These are given in the normalized coordinate [0,1] by :

$$x_i = \frac{i-1}{N-1} \quad (i = 1, 2, \ldots, N) \tag{5}$$

But the Differential Quadrature solutions usually deliver more accurate results with unequally spaced sampling points. A well accepted kind of sampling points using zeros of the orthogonal polynomials in the DQM is the so-called Gauss-Lobatto-Chebyshev points:

$$x_i = \frac{1 - \cos[(i-1)\pi/(N-1)]}{2} \quad (i = 1, 2, \ldots, N) \tag{6}$$

Recent developments of the DQM

1) Usually, the fourth-order differential equations in structural mechanics such as beam and plate's displacement, buckling and free-vibration analysis have two boundary equations at each boundary.

2)Two conditions at the same point provoke a big and real challenge for the classic Differential Quadrature Method, because in the classic DQM we have only one quadrature equation at one point but two boundary equations are to be implemented.

Therefore, Bert and Malik (1996), Jang et al. (1989),Kang et al. (1995), Kukreti et al. (1992) and Striz et al.(1988) proposed the δ-type grid arrangements, that is, besides the two boundary points, two additional adjacent points with an order of 10^{-5} distance to the boundary points were also treated as boundary points. Therefore, there are two boundary points at each boundary corresponding to their two respective boundary conditions.

δ-type grid arrangement deficiencies :

1) In solid dynamics problems, one has two initial conditions at the initial time, that is the initial displacement and initial velocity. The same problems (two conditions at the same point) were also encountered. Therefore no one

paper has appeared about solid dynamics problems solved by the classic DQM.

2) Although the δ-type grid arrangements work well for some circumstances, this type of boundary grid arrangement is not mathematically suitable and will sometimes cause ill-conditioned problems.

The improved proposed Differential Quadrature Method

This paper will propose an Improved Differential Quadrature Method using Hermite interpolation functions to apply the multiple boundary or initial conditions exactly without using the δ-point technique with unequally spaced sampling points .

The independent variables are chosen to be the function value and its derivatives of possible lowest order wherever necessary.

Consider a one-dimensional variable $\psi(x)$ in domain of $x_1 < x < x_n$ and divide the domain by N points x_i (i = 1,2,…,N).

Let n_i denote the number of equations corresponding to the point x_i.

Using Hermite interpolation functions:

$$\psi(x) = \sum_{j=1}^{N} \left(h_{j0}(x)\psi_j^{(0)} + h_{j1}(x)\psi_j^{(1)} + \cdots + h_{j(n_j-1)}(x)\psi_j^{(n_j-1)} \right)$$

Properties of Hermite interpolation shape functions when $x = x_j$

$h_{j0}^{(i)}(x_j)$		$h_{j1}^{(i)}(x_j)$		......	$h_{j(n_j-1)}^{(i)}(x_j)$	
$h_{j0}(x_j)$	1	$h_{j1}(x_j)$	0	......	$h_{j(n_j-1)}(x_j)$	0
$h_{j0}^{(1)}(x_j)$	0	$h_{j1}^{(1)}(x_j)$	1	......	$h_{j(n_j-1)}^{(2)}(x_j)$	0
......	0	......	0	......	......	0
$h_{j0}^{(n_j-1)}(x_j)$	0	$h_{j1}^{(n_j-1)}(x_j)$	0	......	$h_{j(n_j-1)}^{(n_j-1)}(x_j)$	1

$$\psi(x) = \sum_{j=1}^{N}\left(h_{j0}(x)\psi_j^{(0)} + h_{j1}(x)\psi_j^{(1)} + \cdots + h_{j(n_j-1)}(x)\psi_j^{(n_j-1)}\right) = \sum_{k=1}^{M} h_k(x)U_k$$

$$M = \sum_{j=1}^{N} n_j,$$

$$\begin{aligned}\{U_k\} &= \{U_1, U_2, \ldots, U_M\} \\ &= \{\psi_1^{(0)}, \psi_1^{(1)}, \ldots, . \\ &\quad \psi_1^{(n_1-1)}, \ldots, \psi_N^{(0)}, \psi_N^{(1)} \ldots, \psi_N^{(n_N-1)}\},\end{aligned}$$

$$\begin{aligned}\{h_k\} &= \{h_1, h_2, \ldots, h_M\}^{\mathrm{T}} \\ &= \{h_{10}(x), h_{11}(x), \ldots, h_{1(n_1-1)}(x), \ldots, h_{N0}(x), \\ &\quad h_{N1}(x), \ldots, h_{N(n_N-1)}(x)\}^{\mathrm{T}}\end{aligned}$$

Expression of the newly proposed DQM

$$\frac{\mathrm{d}^r \psi(x_i)}{\mathrm{d}x^r} = \sum_{k=1}^{M} E_{ik}^{(r)} U_k \qquad (i = 1, 2, \ldots, N)$$

Where $E_{ik}^{(r)}$ are called the weighting coefficients of the rth-order derivative of the function at point x_i.

$$E_{ik}^{(r)} = h_k^{(r)}(x_i)$$

$$= \left\{ h_{10}^{(r)}(x_i), h_{11}^{(r)}(x_i), \ldots, h_{1(n_1-1)}^{(r)}(x_i), \ldots, h_{N0}^{(r)}(x_i), h_{N1}^{(r)}(x_i), \ldots, h_{N(n_N-1)}^{(r)}(x_i) \right\}$$

For points with more than one equation such as single span beam's end points, the more than one independent variable is introduced to implement the same number of equations. Then the deficiencies of the δ-type grid arrangements are eliminated, and the boundary conditions are applied directly.

In conclusion, the calssic DQM has only the function values as the independent variables. Therefore at one point only one differential quadrature analog can be implemented.

But in the newly proposed Differential Quadrature one has the function value and its derivatives wherever necessary as the independent variables.

Thus at one point, more than one differential quadrature analog can be implemented in the proposed Differential Quadrature. The resulting weighting coefficient of the classic DQM is a matrix of N × N. But the resulting weighting coefficient of the proposed Differential Quadrature is a matrix of N × M.

Single-span Bernoulli-Euler Beam's Buckling Analysis

The governing equation of single-span Bernoulli-Euler beam's buckling problem is :

$$\frac{d^4 w}{dx^4} - \frac{P}{EI}\frac{d^2 w}{dx^2} = 0 \qquad x \subset [0, L] = [0, 1]$$

where ω is the displacement function in the y direction,
E and I denote the modulus of elasticity and
principal moment of inertia about the z-axis, respectively.
P is the compressive axial load.

The single-span Bernoulli-Euler beam has four boundary conditions, two at each end. The beam is divided into N-1 sections using nonequally spaced points. The boundary conditions are usually the following forms in the buckling analysis:

$$w_i = 0; \quad w_i^{(1)} = 0; \quad EIw_i^{(2)} = 0;$$

$$EIw_i^{(3)} = 0 \qquad (i = 1 \text{ or } N)$$

In this example, there are two boundary points x_1 and x_N. At point x_1 there are two boundary conditions and thus two independent variables w_1 and $w_1^{(1)}$. At point x_N there are also two boundary conditions and then two independent variables w_N and $w_N^{(1)}$.

Therefore:

$$n_1 = n_N = 2,$$

$$n_2 = n_3 = \cdots = n_{N-1} = 1,$$

$$M = \sum_{j=1}^{N} n_j = N + 2$$

Differential Quadrature expression

$$w^{(r)}(x_i) = \sum_{j=1}^{N+2} E_{ij}^{(r)} U_j$$

where

$$\{E_{ij}^{(r)}\} = \{h_{10}^{(r)}(x_i), h_{11}^{(r)}(x_i), h_{20}^{(r)}(x_i), \ldots, h_{N0}^{(r)}(x_i), h_{N1}^{(r)}(x_i)\}$$

$$= \{h_1^{(r)}(x_i), h_2^{(r)}(x_i), \ldots, h_{N+1}^{(r)}(x_i), h_{N+2}^{(r)}(x_i)\}$$

$$\{U_j\} = \{w_1, w_1^{(1)}, w_2, \ldots, w_N, w_N^{(1)}\}$$

$$= \{U_1, U_2, \ldots, U_{N+2}\}$$

Interpolation shape function

Since we have two boundary conditions at the first point, two interpolation functions $h_{10}(x)$ & $h_{11}(x)$ are defined for this point as follows:

$$h_{10}(x) = (a_1 x^2 + b_1 x + c_1) l_1(x)$$

where $h_{10}(x)$ has the following properties:

$$h_{10}(x_1) = 1; \quad h_{10}^{(1)}(x_1) = 0; \quad h_{10}^{(1)}(x_N) = 0;$$
$$h_{10}(x_j) = 0, \qquad (j = 2, 3, \ldots, N)$$

Notice that $l_1(x_N) = 0,\ l_1(x_1) = 1$

$$\Rightarrow \begin{cases} (a_1 x_1^2 + b_1 x_1 + c_1) l_1(x_1) = a_1 x_1^2 + b_1 x_1 + c_1 = 1 \\ (2a_1 x_1 + b_1) l_1(x_1) + (a_1 x_1^2 + b_1 x_1 + c_1) l_1^{(1)}(x_1) = 0 \\ (2a_1 x_N + b_1) l_1(x_N) + (a_1 x_N^2 + b_1 x_N + c_1) l_1^{(1)}(x_N) = 0 \end{cases}$$

$$\Rightarrow \begin{aligned} a_1 &= \frac{-1}{(x_1 - x_N)^2} + \frac{-l_1^{(1)}(x_1)}{(x_1 - x_N)} \\ b_1 &= \frac{1}{(x_1 - x_N)} - a_1(x_1 + x_N) \\ &= \frac{2x_1}{(x_1 - x_N)^2} + \frac{(x_1 + x_N) l_1^{(1)}(x_1)}{(x_1 - x_N)} \\ c_1 &= 1 - a_1 x_1^2 - b_1 x_1 \\ &= \frac{(x_N - 2x_1) x_N}{(x_1 - x_N)^2} + \frac{-x_1 x_N l_1^{(1)}(x_1)}{(x_1 - x_N)} \end{aligned}$$

$$h_{11}(x) = \left(a_{11}x^2 + b_{11}x + c_{11}\right) l_1(x)$$

The $h_{11}(x)$ should have the following properties:

$$h_{11}^{(1)}(x_1) = 1; \; h_{11}^{(1)}(x_N) = 0;$$
$$h_{11}(x_j) = 0, \quad (j = 1, 2, \ldots, N)$$

Where

$$a_{11} = \frac{1}{x_1 - x_N}; \quad b_{11} = \frac{-(x_1 + x_N)}{x_1 - x_N}; \quad c_{11} = \frac{x_1 x_N}{x_1 - x_N}$$

$$h_{j0}(x) = (a_j x^2 + b_j x + c_j) l_j(x)$$

$$(j = 2, 3, \ldots, N-1)$$

Where its properties are as follows:

$$h_{j0}(x_j) = 1; \quad h_{j0}^{(1)}(x_1) = 0; \quad h_{j0}^{(1)}(x_N) = 0; \quad h_{j0}(x_i) = 0$$

$$(i = 1, 2, \ldots, N; j = 2, 3, \ldots, N-1; \quad i \neq j)$$

$$\Rightarrow \quad \begin{aligned} a_j &= \frac{1}{x_j^2 - x_j(x_1 + x_N) + x_1 x_N} \\ b_j &= \frac{-(x_1 + x_N)}{x_j^2 - x_j(x_1 + x_N) + x_1 x_N} \\ c_j &= \frac{x_1 x_N}{x_j^2 - x_j(x_1 + x_N) + x_1 x_N} \end{aligned}$$

$$h_{N0}(x) = (a_N x^2 + b_N x + c_N) l_N(x)$$

Where its properties are as follows:

$$h_{N0}(x_N) = 1; \quad h_{N0}^{(1)}(x_N) = 0; \quad h_{N0}^{(1)}(x_1) = 0;$$
$$h_{N0}(x_i) = 0 \qquad (i = 1, 2, \ldots, N-1)$$

$$\Rightarrow$$

$$\begin{aligned}
a_N &= \frac{-1}{(x_1 - x_N)^2} + \frac{l_N^{(1)}(x_N)}{(x_1 - x_N)} \\
b_N &= \frac{-1}{(x_1 - x_N)} - a_N(x_1 + x_N) \\
&= \frac{2x_N}{(x_1 - x_N)^2} - \frac{(x_1 + x_N) l_N^{(1)}(x_N)}{(x_1 - x_N)} \\
c_N &= 1 - a_N x_N^2 - b_N x_N \\
&= \frac{(x_1 - 2x_N)x_1}{(x_1 - x_N)^2} + \frac{x_1 x_N l_N^{(1)}(x_N)}{(x_1 - x_N)}
\end{aligned}$$

$$h_{N1}(x) = \left(a_{N1}x^2 + b_{N1}x + c_{N1}\right)l_N(x)$$

Where its properties are as follows:

$$h_{N1}^{(1)}(x_N) = 1; \quad h_{N1}^{(1)}(x_1) = 0;$$
$$h_{N1}(x_j) = 0, \qquad (j = 1, 2, \ldots, N)$$

$$\Rightarrow \qquad a_{N1} = \frac{-1}{x_1 - x_N}; \quad b_{N1} = \frac{x_1 + x_N}{x_1 - x_N}; \quad c_{N1} = \frac{-x_1 x_N}{x_1 - x_N}$$

Explicit weighting coefficients

$$\frac{d^4 w}{dx^4} - \frac{P}{EI}\frac{d^2 w}{dx^2} = 0 \qquad x \subset [0, L] = [0, 1]$$

$$w^{(r)}(x_i) = \sum_{j=1}^{N+2} E_{ij}^{(r)} U_j$$

$$E_{ij}^{(r)} = \frac{d^r}{dx^r} h_j(x_i)$$

$$(i = 1, 2, \ldots, N;\ j = 1, 2, \ldots, N+2;\ r = 1, 2, 3, 4)$$

Now the differential quadrature analog of the beam governing equation is:

$$\sum_{j=1}^{N+2} E_{ij}^{(4)} U_j - \lambda \sum_{j=1}^{N+2} E_{ij}^{(2)} U_j = 0$$

$$\lambda = P/EI$$

$$(i = 2, 3, \ldots, N-1)$$

The differential quadrature analogs of the boundary conditions are :

$$w_i = 0; \;\; w_i^{(1)} = 0; \;\; EI \sum_{j=1}^{N+2} E_{ij}^{(2)} U_j = 0;$$

$$EI \sum_{j=1}^{N+2} E_{ij}^{(3)} U_j = 0 \quad (i = 1 \text{ or } N)$$

$$
\begin{bmatrix} [S_{bb}] & [S_{bd}] \\ [S_{db}] & [S_{dd}] \end{bmatrix} \begin{Bmatrix} \{U_b\} \\ \{U_d\} \end{Bmatrix} - \lambda \begin{bmatrix} [0] & [0] \\ [Q_{db}] & [Q_{dd}] \end{bmatrix} \begin{Bmatrix} \{U_b\} \\ \{U_d\} \end{Bmatrix} = 0
$$

Where the subscript **b** indicates the grid points used for writing the quadrature analog of the boundary conditions and the subscript **d** is related to other points .

$$
\{U_b\} = \{U_1, U_2, U_{N+1}, U_{N+2}\} = \left\{w_1, w_1^{(1)}, w_N, w_N^{(1)}\right\}.
$$

$$
\{U_d\} = \{U_3, U_4, \ldots, U_N\} = \{w_2, w_3, \ldots, w_{N-1}\}.
$$

$$\Rightarrow \begin{array}{l} [S_{bb}]\{U_b\} + [S_{bd}]\{U_d\} = 0 \\ [S_{db}]\{U_b\} + [S_{dd}]\{U_d\} - \lambda([Q_{db}]\{U_b\} + [Q_{dd}]\{U_d\}) = 0 \end{array}$$

$$\{U_b\} = -[S_{bb}]^{-1}[S_{bd}]\{U_d\}$$

$$\Rightarrow \begin{array}{l} \left([S_{dd}] - [S_{db}][S_{bb}]^{-1}[S_{bd}]\right)\{U_d\} \\ \quad - \lambda\left([Q_{dd}] - [Q_{db}][S_{bb}]^{-1}[S_{bd}]\right)\{U_d\} = 0 \end{array}$$

$$\Rightarrow \quad ([S] - \lambda[Q])\{U_d\} = 0$$

This is a generalized eigenvalue equation.

By the procedure proposed here, one obtains the normalized critical buckling axial load λ of the beam with various boundary conditions. The calculated λ is compared with analytic results in Table 2. Good agreements were obtained. When more sampling points are employed, Table 2 shows that the convergence rate is very rapid.

Table 2. Comparison of beam normalized critical buckling load λ under various boundary conditions

N	Pinned–Pinned	Fixed–Fixed	Fixed–Pinned
Analytic	9.869604	39.47842	20.19073
6	9.867287	40.44472	20.17477
7	9.869683	39.37706	20.18902
8	9.869631	39.48238	20.19110
9	9.869604	39.47825	20.19075
10	9.869604	39.47845	20.19072
11	9.869604	39.47842	20.19073

تحلیل کمانش تیر برنولی-اولر با در نظر گرفتن 4 نقطه بر روی آن :

$$\frac{d^4 w}{dx^4} - \frac{P}{EI}\frac{d^2 w}{dx^2} = 0 \qquad x \subset [0, L] = [0, 1]$$

$$w^{(r)}(x_i) = \sum_{j=1}^{N+2} E_{ij}^{(r)} U_j \quad \Rightarrow \quad \sum_{j=1}^{6} E^{(4)}{}_{ij} U_j - \lambda \sum_{j=1}^{6} E^{(2)}{}_{ij} U_j = 0$$

$$i = 2 , 3$$

$$M = N + 2 = \sum_{j=1}^{N} n_j = 2 + 1 + 1 + 2 = 6$$

$$N = 4$$

$$i = 1 , 2 , 3 , 4$$

$$x_1 = 0 \; , \quad x_2 = 0.25 \; , \quad x_3 = 0.75 \; , \; x_4 = 1$$

If $i = 2$

$$E_{21}^{(4)}U_1 + E_{22}^{(4)}U_2 + E_{23}^{(4)}U_3 + E_{24}^{(4)}U_4 + E_{25}^{(4)}U_5 + E_{26}^{(4)}U_6$$
$$-\lambda\left[E_{21}^{(2)}U_1 + E_{22}^{(2)}U_2 + E_{23}^{(2)}U_3 + E_{24}^{(2)}U_4 + E_{25}^{(2)}U_5 + E_{26}^{(2)}U_6\right] = 0$$

If $i = 3$

$$E_{31}^{(4)}U_1 + E_{32}^{(4)}U_2 + E_{33}^{(4)}U_3 + E_{34}^{(4)}U_4 + E_{35}^{(4)}U_5 + E_{36}^{(4)}U_6$$
$$-\lambda\left[E_{31}^{(2)}U_1 + E_{32}^{(2)}U_2 + E_{33}^{(2)}U_3 + E_{34}^{(2)}U_4 + E_{35}^{(2)}U_5 + E_{36}^{(2)}U_6\right] = 0$$

تیر مورد نظر در دو انتها دارای تکیه گاه ساده می باشد :

$$\text{Pinned - Pinned} \Rightarrow \begin{cases} w_i = 0 & \text{جابجایی در دو انتها صفر} \\ EI\sum_{j=1}^{6} E_{ij}^{(2)} U_j = 0 & \text{ممان خمشی در دو انتها صفر} \end{cases}$$

$i = 1 \quad \text{or} \quad 4$

$$\text{If } \; i=1 \rightarrow \begin{cases} w_1 = 0 \Rightarrow U_1 = 0 \\ E_{11}^{(2)}U_1 + E_{12}^{(2)}U_2 + E_{13}^{(2)}U_3 + E_{14}^{(2)}U_4 + E_{15}^{(2)}U_5 + E_{16}^{(2)}U_6 = 0 \end{cases}$$

$$\text{If } \; i=4 \rightarrow \begin{cases} w_4 = 0 \Rightarrow U_5 = 0 \\ E_{41}^{(2)}U_1 + E_{42}^{(2)}U_2 + E_{43}^{(2)}U_3 + E_{44}^{(2)}U_4 + E_{45}^{(2)}U_5 + E_{46}^{(2)}U_6 = 0 \end{cases}$$

$$\begin{bmatrix} [S_{bb}] & [S_{bd}] \\ [S_{db}] & [S_{dd}] \end{bmatrix} \begin{Bmatrix} \{U_b\} \\ \{U_d\} \end{Bmatrix} - \lambda \begin{bmatrix} [0] & [0] \\ [Q_{db}] & [Q_{dd}] \end{bmatrix} \begin{Bmatrix} \{U_b\} \\ \{U_d\} \end{Bmatrix} = 0$$

$$\{U_b\} = \{U_1 \quad U_2 \quad U_5 \quad U_6\} = \{w_1 \quad w_1^{(1)} \quad w_4 \quad w_4^{(1)}\}$$

$$\{U_d\} = \{U_3 \quad U_4\} = \{w_2 \quad w_3\}$$

$$\begin{bmatrix} E_{11}^{(2)} & E_{12}^{(2)} & E_{15}^{(2)} & E_{16}^{(2)} \\ E_{41}^{(2)} & E_{42}^{(2)} & E_{45}^{(2)} & E_{46}^{(2)} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_5 \\ U_6 \end{Bmatrix} + \begin{bmatrix} E_{13}^{(2)} & E_{14}^{(2)} \\ E_{43}^{(2)} & E_{44}^{(2)} \end{bmatrix} \begin{Bmatrix} U_3 \\ U_4 \end{Bmatrix} = 0$$

$$\begin{bmatrix} E_{21}^{(4)} & E_{22}^{(4)} & E_{25}^{(4)} & E_{26}^{(4)} \\ E_{31}^{(4)} & E_{32}^{(4)} & E_{35}^{(4)} & E_{36}^{(4)} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_5 \\ U_6 \end{Bmatrix} + \begin{bmatrix} E_{23}^{(4)} & E_{24}^{(4)} \\ E_{33}^{(4)} & E_{34}^{(4)} \end{bmatrix} \begin{Bmatrix} U_3 \\ U_4 \end{Bmatrix}$$

$$-\lambda \left(\begin{bmatrix} E_{21}^{(2)} & E_{22}^{(2)} & E_{25}^{(2)} & E_{26}^{(2)} \\ E_{31}^{(2)} & E_{32}^{(2)} & E_{35}^{(2)} & E_{36}^{(2)} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_5 \\ U_6 \end{Bmatrix} + \begin{bmatrix} E_{23}^{(2)} & E_{24}^{(2)} \\ E_{33}^{(2)} & E_{34}^{(2)} \end{bmatrix} \begin{Bmatrix} U_3 \\ U_4 \end{Bmatrix} \right) = 0$$

بدست آوردن ضرایب وزنی :

$$E_{ij}^{(r)} = \frac{\mathrm{d}^r}{\mathrm{d}x^r} h_j(x_i)$$

$$(i = 1, 2, \ldots, 4\,;\; j = 1, 2, \ldots, 6\,;\; r = 1, 2, 3, 4)$$

$$\left\{\begin{matrix} h_{10}^{(r)} & h_{11}^{(r)} & h_{20}^{(r)} & h_{30}^{(r)} & h_{40}^{(r)} & h_{41}^{(r)} \\ .\downarrow & \ldots\downarrow & \ldots\downarrow & \ldots\downarrow & \ldots\downarrow & \ldots\downarrow \\ h_{1}^{(r)} & h_{2}^{(r)} & h_{3}^{(r)} & h_{4}^{(r)} & h_{5}^{(r)} & h_{6}^{(r)} \end{matrix}\right\}$$

$$E_{12}^{(2)} = \frac{d^2}{dx^2} h_2(x_1)$$

$a_{11} = -1$ $b_{11} = +1$ $c_{11} = 0$

$$h_2(x) = (-x^2 + x) l_1(x)$$

$$l_1(x) = \frac{(x - x_2)(x - x_3)(x - x_4)}{(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)}$$

$$\Rightarrow \quad E_{12}^{(2)} = -14.6667$$

$$E_{16}^{(2)} = \frac{d^2}{dx^2} h_6(x_1)$$

$a_{41} = +1$ $b_{41} = -1$ $c_{41} = 0$

$$h_6(x) = (x^2 - x) l_4(x)$$

$$l_4(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)}$$

$$\Rightarrow \quad E_{16}^{(2)} = -2.00$$

$$E_{42}^{(2)} = \frac{d^2}{dx^2} h_2(x_4)$$

$a_{11} = -1$ $b_{11} = +1$ $c_{11} = 0$

$$h_2(x) = (-x^2 + x) l_1(x)$$

$$\Rightarrow \quad E_{42}^{(2)} = +2.00$$

$$l_1(x) = \frac{(x - x_2)(x - x_3)(x - x_4)}{(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)}$$

$$E_{46}^{(2)} = \frac{d^2}{dx^2} h_6(x_4)$$

$$\Rightarrow \quad E_{46}^{(2)} = +14.6667$$

$$h_6(x) = (x^2 - x) l_4(x)$$

$$E_{13}^{(2)} = \frac{d^2}{dx^2} h_3(x_1)$$

$a_2 = -5.33 \quad b_2 = +5.33 \quad c_2 = 0$

$$h_3(x) = (-5.33x^2 + 5.33x) l_2(x)$$

$$l_2(x) = \frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)}$$

$$\Rightarrow \quad E_{13}^{(2)} = 85.3328$$

$$E_{14}^{(2)} = \frac{d^2}{dx^2} h_4(x_1)$$

$a_3 = -5.33 \quad b_3 = +5.33 \quad c_3 = 0$

$$h_4(x) = (-5.33x^2 + 5.33x) l_3(x) \quad \Rightarrow \quad E_{14}^{(2)} = -28.4443$$

$$E_{43}^{(2)} = \frac{d^2}{dx^2} h_3(x_4) \quad \Rightarrow \quad E_{43}^{(2)} = -28.4443$$

$$h_3(x) = (-5.33x^2 + 5.33x) l_2(x)$$

$$E_{44}^{(2)} = \frac{d^2}{dx^2} h_4(x_4) \quad \Rightarrow \quad E_{44}^{(2)} = 85.3328$$

$$h_4(x) = (-5.33x^2 + 5.33x) l_3(x)$$

$$E_{22}^{(4)} = \frac{d^4}{dx^4} h_2(x_2) \quad \Rightarrow \quad E_{22}^{(4)} = -224$$

$$h_2(x) = (-x^2 + x) l_1(x)$$

$$E_{26}^{(4)} = \frac{d^4}{dx^4} h_6(x_2) \quad \Rightarrow \quad E_{26}^{(4)} = -96$$

$$h_6(x) = (x^2 - x) l_4(x)$$

$$E_{32}^{(4)} = \frac{d^4}{dx^4} h_2(x_3) \quad \Rightarrow \quad E_{32}^{(4)} = 96$$

$$h_2(x) = (-x^2 + x) l_1(x)$$

$$E_{36}^{(4)} = \frac{d^4}{dx^4} h_6(x_3) \quad \Rightarrow \quad E_{36}^{(4)} = 224$$

$$h_6(x) = (x^2 - x) l_4(x)$$

$$E_{23}^{(4)} = \frac{d^4}{dx^4} h_3(x_2) \quad \Rightarrow \quad E_{23}^{(4)} = 2048$$

$$h_3(x) = (-5.33x^2 + 5.33x) l_2(x)$$

$$E_{24}^{(4)} = \frac{d^4}{dx^4} h_4(x_2) \quad \Rightarrow \quad E_{24}^{(4)} = -1365.3$$

$$h_4(x) = (-5.33x^2 + 5.33x) l_3(x)$$

$$E_{33}^{(4)} = \frac{d^4}{dx^4} h_3(x_3) \quad \Rightarrow \quad E_{33}^{(4)} = -1365.3$$

$$h_3(x) = (-5.33x^2 + 5.33x) l_2(x)$$

$$E_{34}^{(4)} = \frac{d^4}{dx^4} h_4(x_3) \quad \Rightarrow \quad E_{34}^{(4)} = 2048$$

$$h_4(x) = (-5.33x^2 + 5.33x) l_3(x)$$

$$E_{22}^{(2)} = \frac{d^2}{dx^2} h_2(x_2) \quad \Rightarrow \quad E_{22}^{(2)} = 0.50$$

$$h_2(x) = (-x^2 + x) l_1(x)$$

$$E_{26}^{(2)} = \frac{d^2}{dx^2} h_6(x_2) \quad \Rightarrow \quad E_{26}^{(2)} = 1.1667$$

$$h_6(x) = (x^2 - x) l_4(x)$$

$$E_{32}^{(2)} = \frac{d^2}{dx^2} h_2(x_3) \quad \Rightarrow \quad E_{32}^{(2)} = -1.1667$$

$$h_2(x) = (-x^2 + x) l_1(x)$$

$$E_{36}^{(2)} = \frac{d^2}{dx^2} h_6(x_3) \quad \Rightarrow \quad E_{36}^{(2)} = -0.50$$

$$h_6(x) = (x^2 - x) l_4(x)$$

$$E_{23}^{(2)} = \frac{d^2}{dx^2} h_3(x_2) \quad \Rightarrow \quad E_{23}^{(2)} = -28.4443$$

$$h_3(x) = (-5.33x^2 + 5.33x) l_2(x)$$

$$E_{24}^{(2)} = \frac{d^2}{dx^2} h_4(x_2) \quad \Rightarrow \quad E_{24}^{(2)} = 21.3332$$

$$h_4(x) = (-5.33x^2 + 5.33x) l_3(x)$$

$$E_{33}^{(2)} = \frac{d^2}{dx^2} h_3(x_3) \quad \Rightarrow \quad E_{33}^{(2)} = 21.3332$$

$$h_3(x) = (-5.33x^2 + 5.33x) l_2(x)$$

$$E_{34}^{(2)} = \frac{d^2}{dx^2} h_4(x_3) \quad \Rightarrow \quad E_{34}^{(2)} = -28.4443$$

$$h_4(x) = (-5.33x^2 + 5.33x) l_3(x)$$

$$E_{11}^{(2)} = \frac{d^2}{dx^2} h_1(x_1)$$

$a_1 = -7.3333$ $b_1 = +8.3333$ $c_1 = 1$

$$h_1(x) = (-7.3333x^2 + 8.3333x + 1) l_1(x) \quad \Rightarrow \quad E_{11}^{(2)} = -98.8889$$

$$E_{15}^{(2)} = \frac{d^2}{dx^2} h_5(x_1)$$

$a_4 = -7.3333$ $b_4 = +8.3333$ $c_4 = 0$

$$h_5(x) = (-7.3333x^2 + 8.3333x) l_4(x) \quad \Rightarrow \quad E_{15}^{(2)} = 16.6667$$

$$E_{41}^{(2)} = \frac{d^2}{dx^2} h_1(x_4) \quad \Rightarrow \quad E_{41}^{(2)} = -8.6667$$

$$h_1(x) = (-7.3333x^2 + 8.3333x + 1) l_1(x)$$

$$E_{45}^{(2)} = \frac{d^2}{dx^2} h_5(x_4) \quad \Rightarrow \quad E_{45}^{(2)} = -73.5556$$

$$h_5(x) = (-7.3333x^2 + 8.3333x) l_4(x)$$

$$E_{21}^{(4)} = \frac{d^4}{dx^4} h_1(x_2) \quad \Rightarrow \quad E_{21}^{(4)} = -1770.7$$

$$h_1(x) = (-7.3333x^2 + 8.3333x + 1) l_1(x)$$

$$E_{25}^{(4)} = \frac{d^4}{dx^4} h_5(x_2) \quad \Rightarrow \quad E_{25}^{(4)} = 832$$

$$h_5(x) = (-7.3333x^2 + 8.3333x) l_4(x)$$

$$E_{31}^{(4)} = \frac{d^4}{dx^4} h_1(x_3) \quad \Rightarrow \quad E_{31}^{(4)} = 576$$

$$h_1(x) = (-7.3333x^2 + 8.3333x + 1) l_1(x)$$

$$E_{35}^{(4)} = \frac{d^4}{dx^4} h_5(x_3) \quad \Rightarrow \quad E_{35}^{(4)} = -1514.7$$

$$h_5(x) = (-7.3333x^2 + 8.3333x) l_4(x)$$

$$E_{21}^{(2)} = \frac{d^2}{dx^2} h_1(x_2) \quad \Rightarrow \quad E_{21}^{(2)} = 16.3333$$

$$h_1(x) = (-7.3333x^2 + 8.3333x + 1) l_1(x)$$

$$E_{25}^{(2)} = \frac{d^2}{dx^2} h_5(x_2) \quad \Rightarrow \quad E_{25}^{(2)} = -10.5556$$

$$h_5(x) = (-7.3333x^2 + 8.3333x) l_4(x)$$

$$E_{31}^{(2)} = \frac{d^2}{dx^2} h_1(x_3) \quad \Rightarrow \quad E_{31}^{(2)} = -11.8889$$

$$h_1(x) = (-7.3333x^2 + 8.3333x + 1) l_1(x)$$

$$E_{35}^{(2)} = \frac{d^2}{dx^2} h_5(x_3) \quad \Rightarrow \quad E_{35}^{(2)} = 17.6667$$

$$h_5(x) = (-7.3333x^2 + 8.3333x) l_4(x)$$

فرم نهایی معادله کمانش تیر برنولی – اویلر به همراه ماتریس ضرایب وزنی

$$\begin{bmatrix} [S_{bb}] & [S_{bd}] \\ [S_{db}] & [S_{dd}] \end{bmatrix} \begin{Bmatrix} \{U_b\} \\ \{U_d\} \end{Bmatrix} - \lambda \begin{bmatrix} [0] & [0] \\ [Q_{db}] & [Q_{dd}] \end{bmatrix} \begin{Bmatrix} \{U_b\} \\ \{U_d\} \end{Bmatrix} = 0$$

$$\begin{bmatrix} -98.8889 & -14.6667 & 16.6667 & -2.0000 & 85.3328 & -28.4443 \\ -8.6667 & 2.00000 & -73.5556 & 14.6667 & -28.4443 & 85.3328 \\ -1770.7 & -224.0000 & 832.000 & -96.0000 & 2048.00 & -1365.3 \\ 576.000 & 96.0000 & -1514.7 & 224.000 & -1365.3 & 2048.00 \end{bmatrix} \begin{Bmatrix} 0 \\ U_2 \\ U_3 \\ U_4 \\ 0 \\ U_6 \end{Bmatrix}$$

$$-\lambda \begin{bmatrix} 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 \\ 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 \\ 16.3333 & 0.5 & -10.5556 & 1.1667 & -28.4443 & 21.3332 \\ -11.8889 & -1.1667 & 17.6667 & -0.5 & 21.3332 & -28.4443 \end{bmatrix} \begin{Bmatrix} 0 \\ U_2 \\ U_3 \\ U_4 \\ 0 \\ U_6 \end{Bmatrix} = 0$$

$$\Rightarrow \begin{array}{l} [S_{bb}]\{U_b\} + [S_{bd}]\{U_d\} = 0 \\ [S_{db}]\{U_b\} + [S_{dd}]\{U_d\} - \lambda\left([Q_{db}]\{U_b\} + [Q_{dd}]\{U_d\}\right) = 0 \end{array}$$

$$\{U_b\} = -[S_{bb}]^{-1}[S_{bd}]\{U_d\}$$

$$\Rightarrow \begin{array}{l} \left([S_{dd}] - [S_{db}][S_{bb}]^{-1}[S_{bd}]\right)\{U_d\} \\ \quad - \lambda\left([Q_{dd}] - [Q_{db}][S_{bb}]^{-1}[S_{bd}]\right)\{U_d\} = 0 \end{array}$$

$$\Rightarrow \quad ([S] - \lambda[Q])\{U_d\} = 0$$

$$\left(\begin{bmatrix} 668.3061 & -2170.1 \\ -2170.1 & 668.3061 \end{bmatrix} - \lambda \begin{bmatrix} -24.2525 & 14.1471 \\ 14.1471 & -24.2525 \end{bmatrix} \right) \begin{Bmatrix} U_3 \\ U_4 \end{Bmatrix} = 0$$

THE END